5.2 (continued)

example $\vec{x}' = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}$

eigenvalues: $\begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$

$$(-1-\lambda)^2 + 4 = 0$$

$$(-1-\lambda)^2 = -4$$

$$(-1-\lambda) = \pm 2i$$

complex conjugate pairs

$$\boxed{\lambda = -1 \pm 2i}$$

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0}$

$\lambda = -1+2i$ $\quad \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} -2 & -2i & 0 \\ -2i & 2 & 0 \end{array}\right]$ multiply row 1 by $-i$
add to row 2

$$\rightarrow \begin{bmatrix} -2 & -2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

repeat w/ $\lambda = -1-2i$

$$\begin{bmatrix} 2i & 2 & 0 \\ -2 & 2i & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

eigenvectors are conjugate pairs

Solutions: $e^{\lambda t}\vec{v}$

$$\lambda = -1+2i, \quad \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = -1-2i, \quad \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

using the top pair:

$$e^{(-1+2i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

want the real-valued solutions to not contain i

$$e^{-t} e^{i(2t)} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Euler's identity

$$e^{it} = \cos(t) + i\sin(t)$$

$$= e^{-t}\left(\cos(2t) + i\sin(2t)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \sin(2t) - i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}_{\text{real part}} + i\,\underbrace{e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}}_{\text{imaginary part}}$$

using the other pair, we get

$$e^{\lambda t}\vec{v} = e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} - i e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$$

Solutions are conjugate pairs

the real and imag parts of either solution
are themselves solutions to $\vec{x}' = A\vec{x}$
so, we use them as fundamental solutions to form
the general solution.

$$\boxed{\vec{x} = c_1 e^{-t}\begin{bmatrix}\sin(2t)\\ \cos(2t)\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-\cos(2t)\\ \sin(2t)\end{bmatrix}}$$

Phase diagram: periodic behavior
spirals (into origin if real part of λ is < 0
away from origin " " > 0)

if real part of λ is 0 → ovals

Spiral direction? clockwise / counterclockwise?

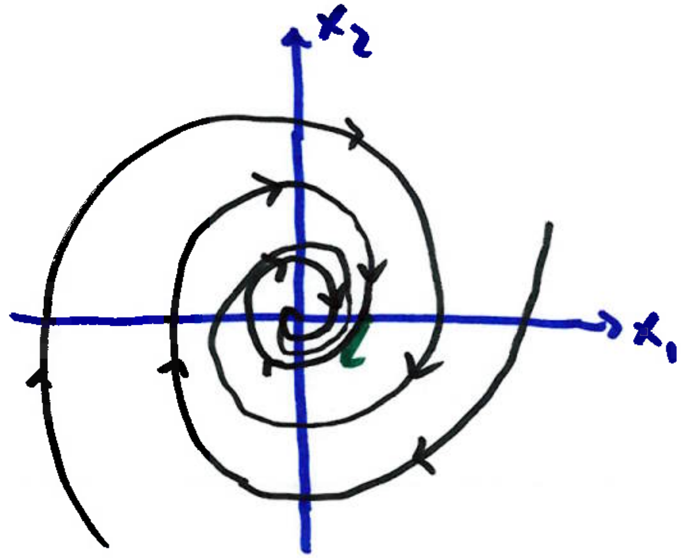
easy way: pick convenient $\vec{x}$ in $\vec{x}' = A\vec{x}$

$$\vec{x}' = \begin{bmatrix}-1 & 2\\ -2 & -1\end{bmatrix}\vec{x}$$

tangent vectors

pick $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$

$\vec{x}' = \begin{bmatrix}-1\\-2\end{bmatrix}$ left and down
clockwise



5.5 Repeated Eigenvalues

$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ $\lambda = 1, 1$ algebraic multiplicity is Two

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

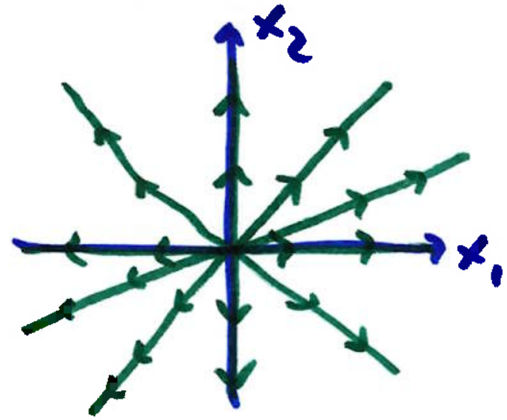
two linearly indp vectors we use as eigenvectors

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ geometric multiplicity is Two

two solutions $e^{\lambda t}\vec{v}$: $e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

general solution: $\vec{x} = c_1 e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

phase diagram:



alg. mult. is two

now let's look at $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\vec{x}$ $\lambda = 2, 2$

$(A - \lambda I)\vec{v} = \vec{0}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

geo. mult. is one

missing one vector

(matrix A is defective

defect of one)

Solution 1: $e^{\lambda t}\vec{v}$ ← generalized eigenvector

Solution 2: $e^{\lambda t}(t\vec{v}+\vec{u})$

where $(A-\lambda I)\vec{u}=\vec{v}$ $\quad \vec{v}=\begin{bmatrix}1\\0\end{bmatrix}$

here, $\begin{bmatrix}0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$ $\quad \vec{u}=\begin{bmatrix}a\\1\end{bmatrix}$ choose any a as long as $\vec{u}\neq\vec{0}$ and is linearly indp from $\vec{v}$

here, $a=0$ $\quad \vec{u}=\begin{bmatrix}0\\1\end{bmatrix}$

general solution:

$$\vec{x}=c_1e^{2t}\begin{bmatrix}1\\0\end{bmatrix}+c_2e^{2t}\left(t\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right)$$